Reinforcement Learning

Deep RL

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June 14, 2026

A Brief History of Deep RL

2013 **DQN** (Deep Q-Network) learns to play Atari games from pixels (Mnih et al., 2013, 2015)

2015 **DDPG** (Deep Deterministic Policy Gradient) (Lillicrap et al., 2015) solves continuous control problems

2015 **TRPO** (Trust-Region Policy Optimization) (Schulman et al., 2015)

2016 **A3C** (Asynchronous Advantage Actor-Critic) (Mnih et al., 2016)
$\rightarrow$ Later superseded by **A2C**, (Synchronous) Advantage Actor-Critic

2016 **AlphaGo** masters the game of Go (Silver et al., 2016) with MCTS + policy networks

2017 **PPO** (Proximal Policy Optimization) (Schulman et al., 2017), another trust-region method

2018 **SAC** (Soft Actor-Critic) (Haarnoja et al., 2018) shows significant improvement in continuous control

2018 DDPG strikes back with **TD3** (Fujimoto et al., 2018)

2022 PPO + **RLHF** behind InstructGPT (Ouyang et al., 2022) and later **ChatGPT**

2023 **REINFORCE** makes a comeback (Ahmadian et al., 2024)

2024 **GRPO** for mathematical reasoning (Shao et al., 2024)

Outline

Outline

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A2C (Advantage Actor-Critic)

- First large-scale implementation of the Actor-Critic architecture (Barto et al., 1983)
- First proposed as **A3C** (**Asynchronous** Advantage Actor-Critic, Mnih et al., 2016)
- "Asynchronous" refers to the fact that many actors (policies) collect experience in parallel and report asynchronously to a central critic
- The synchronous version (**A2C**) turned out to make a more efficient use of GPUs `https://openai.com/index/openai-baselines-acktr-a2c/`

Advantage Estimation

Recall the "advantage" version of the Policy Gradient Theorem

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \frac{1}{1-\gamma} \mathbb{E}\left[\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(A|S) \mathbb{A}^{\pi_{\boldsymbol{\theta}}}(S,A)\right],$$

Where $\mathbb{A}^{\pi_{\boldsymbol{\theta}}}(S,A) = Q^{\pi_{\boldsymbol{\theta}}}(S,A) - V^{\pi_{\boldsymbol{\theta}}}(S)$ is the advantage function.

The main ingredient of A2C is **Advantage Estimation**

- Let $\delta_t = R_{t+1} + \gamma V^{\pi}(S_{t+1}) - V^{\pi}(S_t)$ be the TD (Temporal Difference) error
- Recall that $\mathbb{E}[\delta_t | S_t, A_t] = \mathbb{A}^{\pi}(S_t, A_t)$
- Train a state-critic $v_{\mathbf{w}}(s)$
- Estimate $\mathbb{A}^{\pi}(S_t, A_t)$ as $\delta_t^{\mathbf{w}} = R_{t+1} + \gamma v_{\mathbf{w}}(S_{t+1}) - v_{\mathbf{w}}(S_t)$
- Can be used to compute policy gradient sample $\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(A_t|S_t) \delta_t^{\mathbf{w}}$

Data collection of A2C

At each *iteration* (policy update), A2C collects a *rollout* using the current policy $\pi_{\boldsymbol{\theta}}$.

A rollout can be a single long trajectory or a sequence of variable-length episodes. Let T be the total size of the rollout and τ_t be the final timestep of the episode to which timestep t belongs.

Terminal states must be handled separately:

$$\delta_t^{\mathbf{w}} = \begin{cases} R_{t+1} + \gamma v_{\mathbf{w}}(S_{t+1}) - v_{\mathbf{w}}(S_t) & \text{if } t \neq \tau_t \\ R_{t+1} - v_{\mathbf{w}}(S_t) & \text{otherwise} \end{cases} \tag{1}$$

Critic targets (returns-to-go) can be computed for each $t = 0, \ldots, T-1$ as:

$$G_t = \sum_{k=t}^{\tau_t} \gamma^{k-t} R_{k+1}$$

A2C

Initialize the actor parameters $\boldsymbol{\theta}$ and the critic parameters $\mathbf{w}$

loop for every iteration

Collect a *rollout* of T total timesteps with $\pi_{\boldsymbol{\theta}}$

Compute returns-to-go G_t

Compute TD errors $\delta_t^{\mathbf{w}} = [\![R_{t+1} + \gamma v_{\mathbf{w}}(S_{t+1}) - v_{\mathbf{w}}(S_t)]\!]^*$ based on the current critic $v_{\mathbf{w}}$

Define the actor loss

$$\mathcal{L}_{\text{actor}}(\boldsymbol{\theta}) = -\sum_{t=0}^{T-1} \log \pi_{\boldsymbol{\theta}}(A_t|S_t)\delta_t^{\mathbf{w}}$$

Define the critic loss

$$\mathcal{L}_{\text{critic}}(\mathbf{w}) = \frac{1}{2}\sum_{t=0}^{T-1} \left(v_{\mathbf{w}}(S_t) - G_t\right)^2$$

Minimize $\mathcal{L}_{\text{actor}}(\boldsymbol{\theta}) + \mathcal{L}_{\text{critic}}(\mathbf{w})$ w.r.t. $\boldsymbol{\theta}$ and $\mathbf{w}$ with Adam (gradient descent)

end loop

* brackets denote stop-gradient

On-Policy vs Off-Policy Deep RL

- In **A2C**, the data (rollout) is discarded after each iteration (policy update)
- This makes A2C **on-policy**
- The same holds for the trust-region methods **TRPO, PPO, GRPO** (single-epoch versions)
- In contrast, the replay buffer of **DQN** is used across several iterations (subject to buffer size)
- This makes DQN **off-policy**
- The same holds for **DDPG**, **TD3** and **SAC**

Generalized Advantage Estimation (GAE, Schulman et al., 2016)

Recall k-step returns:

- $v_t^{(1)} = R_{t+1} + \gamma v_{\mathbf{w}}(S_{t+1})$
- $v_t^{(2)} = R_{t+1} + \gamma R_{t+1} + \gamma^2 v_{\mathbf{w}}(S_{t+1})$
- ...
- $v_t^{(k)} = R_{t+1} + \cdots + \gamma^{k-1} R_{t+k} + \gamma^k v_{\mathbf{w}}(S_{t+1})$

Recall the λ-return ($0 < \lambda < 1$):

$$v_t^{\lambda} = \sum_{k=1}^{\infty} \lambda^{k-1} v_t^{(k)}$$

Replace the 1-step return with the λ-step return in advantage estimation:

$$\begin{aligned} \widehat{\mathbb{A}}_t^{\mathrm{GAE}} &= v_t^{\lambda} - v_{\mathbf{w}}(S_t) \\ &= \sum_{k=0}^{\infty} (\gamma\lambda)^k \delta_{t+k}^{\mathbf{w}} \qquad \text{(exercise: can you prove this equality?)} \end{aligned}$$

Like in TD(λ), λ controls the bias-variance trade-off

Data collection of A2C + GAE

At each *iteration* (policy update), A2C collects a *rollout* using the current policy $\pi_{\boldsymbol{\theta}}$.

A rollout can be a single long trajectory or a sequence of variable-length episodes. Let T be the total size of the rollout and τ_t be the final timestep of the episode to which timestep t belongs.

Advantage estimates can be computed for each $t = 0, \ldots, T-1$ as:

$$\widehat{\mathbb{A}}_t^{\mathrm{GAE}} = \sum_{k=t}^{\tau_t} (\gamma\lambda)^{k-t} \delta_k^{\mathbf{w}}$$

Critic targets (λ-returns) can be obtained simply as:

$$v_t^{\lambda} = \widehat{\mathbb{A}}_t^{\mathrm{GAE}} + v_{\mathbf{w}}(S_t) \tag{2}$$

A2C + GAE

Initialize the actor parameters $\boldsymbol{\theta}$ and the critic parameters $\mathbf{w}$

loop for every iteration

Collect a *rollout* of T total timesteps with $\pi_{\boldsymbol{\theta}}$

Compute TD errors $\delta_t^{\mathbf{w}} = [\![R_{t+1} + \gamma v_{\mathbf{w}}(S_{t+1}) - v_{\mathbf{w}}(S_t)]\!]$ based on the current critic $v_{\mathbf{w}}$

Compute advantage estimates $\widehat{\mathbb{A}}_t^{\mathrm{GAE}}$ and TD(λ) targets v_t^{λ}

Define the actor loss

$$\mathcal{L}_{\mathrm{actor}}(\boldsymbol{\theta}) = -\sum_{t=0}^{T-1} \log \pi_{\boldsymbol{\theta}}(A_t|S_t)\widehat{\mathbb{A}}_t^{\mathrm{GAE}}$$

Define the critic loss

$$\mathcal{L}_{\mathrm{critic}}(\mathbf{w}) = \frac{1}{2}\sum_{t=0}^{T-1} \left(v_{\mathbf{w}}(S_t) - v_t^{\lambda}\right)^2$$

Minimize $\mathcal{L}_{\mathrm{actor}}(\boldsymbol{\theta}) + \mathcal{L}_{\mathrm{critic}}(\mathbf{w})$ w.r.t. $\boldsymbol{\theta}$ and $\mathbf{w}$ via gradient descent

end loop

Outline

Deterministic Policy Gradient Theorem

- The policy gradient theorem only holds for **stochastic policies** (Sutton et al., 1999)

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \frac{1}{1-\gamma} \mathbb{E}_{\substack{S \sim d_{\pi_{\boldsymbol{\theta}}} \\ A \sim \pi_{\boldsymbol{\theta}}(\cdot|S)}} \left[\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(A|S) Q^{\pi_{\boldsymbol{\theta}}}(S, A) \right]$$

- Extended to **deterministic policies** by Silver et al. (2014)

Theorem (Deterministic Policy Gradient Theorem (Silver et al., 2014))

Let $\mu_{\boldsymbol{\theta}}(s)$ be a **deterministic** *policy* **differentiable** *in $\boldsymbol{\theta}$ and suppose that the Q-function is* **differentiable in the action** *argument. Then, it holds that:*

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \frac{1}{1-\gamma} \mathbb{E}_{S \sim d^{\mu_{\boldsymbol{\theta}}}} \left[\nabla_{\boldsymbol{\theta}} \mu_{\boldsymbol{\theta}}(S) \nabla_a Q^{\mu_{\boldsymbol{\theta}}}(S, a)|_{a=\mu_{\boldsymbol{\theta}}(S)} \right]$$

Deterministic Policy Gradient

- Deterministic Policy Gradient (DPG) is naturally actor-critic because it needs a critic $q_{\mathbf{w}}(s, a)$ that is **differentiable** w.r.t. a
- DPG is inherently **off-policy** because, the target policy being deterministic, it needs an explicit exploration mechanism
- Typically a **noisy policy** is used for exploration

$$A_t = \mu_{\boldsymbol{\theta}}(S_t) + \epsilon$$

where ϵ is noise sampled from a suitable distribution (e.g. zero-mean Gaussian)

(Deep) Deterministic Policy Gradient

Deep Deterministic Policy Gradient (DDPG): DPG with deep neural netwroks as policy and value approximators (Lillicrap et al., 2015)

- Replay buffer $\mathcal{B}$ to break dependencies
- Target network for both policy $\mu_{\boldsymbol{\theta}^-}(s)$ and value function $q_{\mathbf{w}^-}(s, a)$

DDPG Algorithm

Initialize the replay buffer $\mathcal{B}$ with capacity N
Initialize the actor parameters $\boldsymbol{\theta}$ and the critic parameters $\mathbf{w}$
Initialize the target actor parameters $\boldsymbol{\theta}^- \leftarrow \boldsymbol{\theta}$ and the target critic parameters $\mathbf{w}^- \leftarrow \mathbf{w}$
loop for every iteration
Observe the state S and select action $A = \mu_{\boldsymbol{\theta}}(S) + \epsilon$ where $\epsilon \sim \mathcal{N}$ is noise
Execute A and observe the next state S' and reward R
Store (S, A, R, S') in the replay buffer $\mathcal{B}$
Sample a random minibatch of transitions $(S_j, A_j, R_{j+1}, S_{j+1})$ from the replay buffer $\mathcal{B}$

Compute the targets

$$y_j = \begin{cases} R_{j+1} & \text{if } S_j \text{ is a terminal state} \\ R_{j+1} + \gamma q_{\mathbf{w}^-}(S_{j+1}, \mu_{\boldsymbol{\theta}^-}(S_{j+1})) & \text{otherwise} \end{cases}$$

Perform a gradient descent step on $(y_j - q_{\mathbf{w}}(S_j, A_j))^2$ w.r.t. the critic parameters $\mathbf{w}$
Perform a gradient ascent step on $q_{\mathbf{w}}(S_j, \mu_{\boldsymbol{\theta}}(S_j))$ w.r.t. the actor parameters $\boldsymbol{\theta}$
Update target networks $\boldsymbol{\theta}^- \leftarrow \rho\boldsymbol{\theta}^- + (1-\rho)\boldsymbol{\theta}$ and $\mathbf{w}^- \leftarrow \rho\mathbf{w}^- + (1-\rho)\mathbf{w}$
end loop

Twin Delayed Deep Deterministic Policy Gradient (TD3)

TD3 (Fujimoto et al., 2018) builds on DDPG introducing three main improvements:

- *Twin Critics (inspired by Double Q-Learning) to tackle overestimation bias*
  Train two critics $q^1_{\mathbf{w}}, q^2_{\mathbf{w}}$ (with their respective target networks $q^1_{\mathbf{w}^-}, q^2_{\mathbf{w}^-}$) and compute TD targets

$$y_j = \begin{cases} R_{j+1} & \text{if } S_j \text{ is terminal state} \\ R_{j+1} + \gamma \min_{i \in \{1,2\}} q^i_{\mathbf{w}^-}(S_{j+1}, \mu_{\boldsymbol{\theta}^-}(S_{j+1})) & \text{otherwise} \end{cases}$$

- *Delayed policy updates to reduce instability*
  Update the actor and both target networks at a lower frequency (every k iterations)
- *Target policy smoothing (inspired by Expected SARSA) to avoid overfitting by the deterministic policy*
  Add noise to actions in the target ($\epsilon \sim$ clipped Gaussian)

$$y_j = \begin{cases} R_{j+1} & \text{if } S_j \text{ is terminal state} \\ R_{j+1} + \gamma \min_{i \in \{1,2\}} q^i_{\mathbf{w}^-}(S_{j+1}, \mu_{\boldsymbol{\theta}^-}(S_{j+1}) + \epsilon) & \text{otherwise} \end{cases}$$

Outline

Entropy-Regularized RL

- **Entropy-regularized (or soft)** reward function (Nachum et al., 2017; Haarnoja et al., 2018)

$$r(s,a) + \lambda H(\pi(\cdot|s))$$

  where $\lambda > 0$ is the regularization parameter
- $H(\pi(\cdot|s))$ is the **entropy**

$$H(\pi(\cdot|s)) = -\sum_a \pi(a|s) \log \pi(a|s)$$

- The (differential Shannon) **entropy** is an index of uncertainty
  - Large $H(\pi(\cdot|s)) \to \pi(\cdot|s)$ "very stochastic"
  - Small $H(\pi(\cdot|s)) \to \pi(\cdot|s)$ "almost deterministic"
- Thus, we are encouraging π to be **stochastic**
- This "entropy bonus" can be added to most policy gradient algorithms

Entropy-Regularized RL

- Given a policy π, the **soft value functions** become

$$q_\pi^{\text{soft}}(s,a) \coloneqq r(s,a) + \mathbb{E}\left[\sum_{t=1}^{+\infty} \gamma^t \Big(r(S_t, A_t) + \lambda H(\pi(\cdot|S_t)) \Big) | S_0 = s, A_0 = a, \pi \right]$$

$$v_\pi^{\text{soft}}(s) \coloneqq \mathbb{E}_{A\sim\pi(\cdot|s)}\left[q_\pi^{\text{soft}}(s,A)\right] + \lambda H(\pi(\cdot|S_t))$$

  - When $\lambda \to 0$ they become the standard value functions

- The **greedy softmax policy** w.r.t. function $Q^{\text{soft}}(s,a)$ is defined as

$$\pi^{\text{soft}}(\cdot|s) = \arg\max_\pi \left\{ \mathbb{E}_{A\sim\pi(\cdot|s)}[q^{\text{soft}}(s,A)] + \lambda H(\pi(\cdot|s)) \right\} = \frac{\exp\left(\frac{q^{\text{soft}}(s,\cdot)}{\lambda}\right)}{\sum_a \exp\left(\frac{q^{\text{soft}}(s,a)}{\lambda}\right) \mathrm{d}a}$$

  - When $\lambda \to 0$ it becomes the **greedy** policy

Entropy-Regularized RL

- The state-value function induced by the greedy softmax policy π^{soft} w.r.t. q^{soft} is given by

$$\begin{aligned} v^{\text{soft}}(s) &= \mathbb{E}_{A\sim\pi^{\text{soft}}(\cdot|s)}[q^{\text{soft}}(s,A)] + \lambda H(\pi^{\text{soft}}(\cdot|s)) \\ &= \lambda \log \sum_a \exp\left(\frac{q^{\text{soft}}(s,a)}{\lambda}\right) \mathrm{d}a \end{aligned}$$

- We can rewrite the greedy softmax policy as

$$\pi^{\text{soft}}(a|s) = \exp\left(\frac{q^{\text{soft}}(s,a) - v^{\text{soft}}(s)}{\lambda}\right)$$

Theorem (Policy Improvement Theorem (Haarnoja et al., 2018))

Let $q_\pi^{soft}(s,a)$ be the soft Q-function of policy π. Let π^{soft} be the greedy softmax policy w.r.t. $q_\pi^{soft}(s,a)$. Then, it holds that:

$$q_{\pi^{soft}}^{soft}(s,a) \geq q_\pi^{soft}(s,a) \qquad \forall (s,a) \in \mathcal{S}\times\mathcal{A}$$

Soft Actor-Critic

- Designed for **continuous actions**
- **Actor-critic** approach in the entropy-regularized setting
  - The **actor** approximates the **greedy softmax policy** with a continuous-action (e.g., Gaussian) policy
  - The **critic** estimates the **soft Q-function** $q_{\mathbf{w}}(s, a)$
- The policy update is performed via KL **minimization**

$$\boldsymbol{\theta}' \in \arg\min_{\boldsymbol{\theta} \in \mathbb{R}^d} \mathbb{E}_{S \sim \mathcal{B}} \left[D_{\mathsf{KL}} \left(\pi_{\boldsymbol{\theta}}(\cdot|S) \middle\| \frac{\exp\left(\frac{q_{\mathbf{w}}(S,\cdot)}{\lambda}\right)}{\sum_a \exp\left(\frac{q_{\mathbf{w}}(S,a)}{\lambda}\right) \mathrm{d}a} \right) \right]$$

where $D_{\mathrm{KL}}(p\|q) = \sum_a p(a) \log \frac{p(a)}{q(a)}$ is the Kullback-Liebler divergence (measure of statistical distance) between distributions p and q

Soft Actor-Critic Algorithm

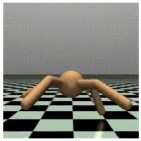
Initialize the replay buffer $\mathcal{B}$ with capacity N
Initialize the actor parameters $\boldsymbol{\theta}$ randomly
Initialize the critic parameters $\mathbf{w}$ randomly
Initialize the target critic parameters $\mathbf{w}^- \leftarrow \mathbf{w}$
loop for every iteration
Observe the state S and select action $A \sim \pi_{\boldsymbol{\theta}}(\cdot|S)$
Execute A and observe the next state S' and reward R
Store (S, A, R, S') in the replay buffer $\mathcal{B}$

Sample a random minibatch of transitions $(S_j, A_j, R_{j+1}, S_{j+1})$ from the replay buffer $\mathcal{B}$
Compute the targets

$$y_j = \begin{cases} R_{j+1} & \text{if } S_j \text{ is terminal state} \\ R_{j+1} + \gamma q_{\mathbf{w}^-}(S_{j+1}, A') - \lambda \log \pi_{\boldsymbol{\theta}}(A'|S_{j+1}), \quad A' \sim \pi_{\boldsymbol{\theta}}(\cdot|S_{j+1}) & \text{otherwise} \end{cases}$$

Perform a gradient descent step on $(y_j - q_{\mathbf{w}}(S_j, A_j))^2$ w.r.t. the critic parameters $\mathbf{w}$
Perform a gradient descent step on D_{KL} w.r.t. the actor parameters $\boldsymbol{\theta}$
Update the target network $\mathbf{w}^- \leftarrow \rho \mathbf{w}^- + (1-\rho)\mathbf{w}$
end loop

Mujoco Locomotion Tasks



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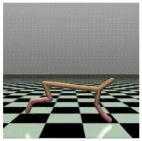
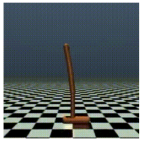


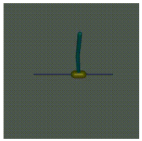
Half Cheetah



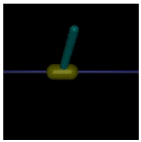
Hopper

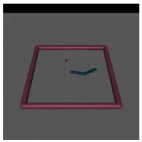
Humanoid

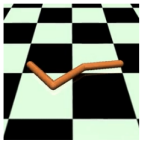
Humanoid Standup



Inverted Double Pendulum



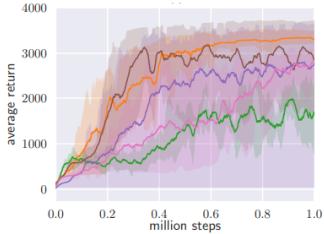




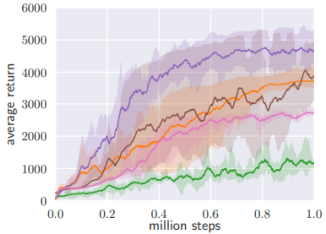
A common benchmark for continuous control using Deep RL

`https://gymnasium.farama.org/environments/mujoco/`

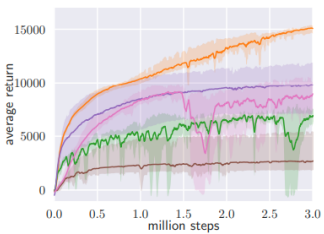
SAC - Experimental Results



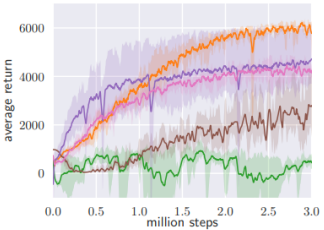
(a) Hopper-v1



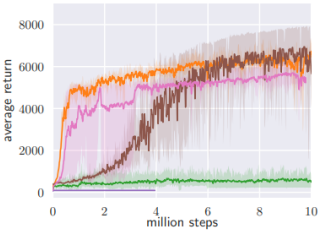
(b) Walker2d-v1



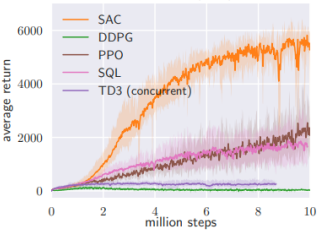
(c) HalfCheetah-v1



(d) Ant-v1



(e) Humanoid-v1



(f) Humanoid (rllab)

Pictures from (Haarnoja et al., 2018)

Outline

Outline

Importance Sampling

- Data $X \sim q$
- Want to estimate $\mu = \mathbb{E}_{x \sim p}[f(x)]$ with $p \neq q$
- **Importance-weighted** estimate

$$\widehat{\mu} = \frac{p(X)}{q(X)} f(X)$$

- **Unbiased**

$$\mathbb{E}[\widehat{\mu}] = \sum_x q(x) \frac{p(x)}{q(x)} f(x) = \mathbb{E}_{x \sim p}[f(x)] = \mu$$

- **Variance** can be large if p and q are very different!

Sample Reuse

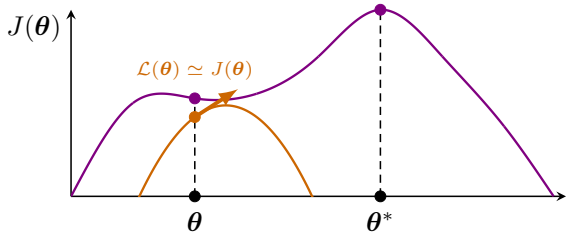
- We want to use samples collected with some policy to evaluate other candidate policies
- We could use **importance weighting** but **variance** might explode
- Use a **surrogate loss** function $\mathcal{L}$ that might be **biased**, but with smaller variance

True loss

$J(\boldsymbol{\theta})$

Surrogate objective

$\mathcal{L}(\boldsymbol{\theta})$



- A surrogate loss is good if

$$\arg\max_{\boldsymbol{\theta} \in \mathbb{R}^d} J(\boldsymbol{\theta}) \approx \arg\max_{\boldsymbol{\theta} \in \mathbb{R}^d} \mathcal{L}(\boldsymbol{\theta})$$

Performance Improvement

Recall once again the Performance Difference Lemma:

Theorem (Performance Difference Lemma, Kakade and Langford, 2002; Howard, 1960)

Given two policies π and π':

$$J(\pi') - J(\pi) = \mathbb{E}_{\substack{S \sim d_{\pi'} \\ A \sim \pi'(\cdot|S)}} [\mathbb{A}^{\pi}(S, A)]$$

Surrogate Loss Construction

- We would like to **maximize** $J(\pi')$ w.r.t. π'
- This is equivalent to maximize the **performance improvement** as π is fixed

$$J(\pi') - J(\pi) = \mathbb{E}_{\substack{S\sim d_{\pi'} \\ A\sim\pi'(\cdot|S)}} [\mathbb{A}^{\pi}(S, A)]$$

- Suppose we have samples from policy π, the expectations are w.r.t. to the new policy π'
- We could use **importance weighting** as

$$J(\pi') - J(\pi) = \mathbb{E}_{\substack{S\sim d_{\pi} \\ A\sim\pi(\cdot|S)}} \left[\frac{d_{\pi'}(S)\pi'(A|S)}{d_{\pi}(S)\pi(A|S)} \mathbb{A}^{\pi}(S, A) \right]$$

- But we don't know $d_{\pi'}$ nor d_{π}. Also, we might introduce a lot of variance in the estimators.

Surrogate Loss Construction

- Instead, we derive a **lower bound** to the performance improvement

$$J(\pi') - J(\pi) = \underbrace{\mathbb{E}_{\substack{S\sim d_\pi \\ A\sim\pi'(\cdot|S)}} \left[\mathbb{A}^\pi(S,A)\right]}_{\text{(A)}} - \underbrace{\sum_s \left(d_\pi(s) - d_{\pi'}(s)\right) \mathbb{E}_{A\sim\pi'(\cdot|s)} \left[\mathbb{A}^\pi(s,A)\right]}_{\text{(B)}}$$

- For (A), we use **importance weighting** but on the policy only

$$\text{(A)} = \mathbb{E}_{\substack{S\sim d_\pi \\ A\sim\pi(\cdot|S)}} \left[\frac{\pi'(A|S)}{\pi(A|S)} \mathbb{A}^\pi(S,A)\right]$$

- For (B), we bound with the **total variation distance** of the policies (Pirotta et al., 2013)

$$\text{(B)} \leq \frac{2\gamma\epsilon}{(1-\gamma)^2} \mathbb{E}_{S\sim d_\pi} \left[D_{\text{TV}}\left(\pi'(\cdot|S), \pi(\cdot|S)\right)\right]$$

where

$$\epsilon \coloneqq \sup_{s\in\mathcal{S}} \left|\mathbb{E}_{A\sim\pi'(\cdot|s)} \left[\mathbb{A}^\pi(s,A)\right]\right| \qquad D_{\text{TV}}\left(\pi'(\cdot|s), \pi(\cdot|s)\right) \coloneqq \frac{1}{2} \sum_a \left|\pi'(\cdot|s) - \pi(\cdot|s)\right|$$

Surrogate Loss

- The resulting **surrogate loss** is given by

$$\mathcal{L}(\pi') \coloneqq \mathbb{E}_{\substack{S\sim d_\pi \\ A\sim\pi(\cdot|S)}} \left[\frac{\pi'(A|S)}{\pi(A|S)}\mathbb{A}^\pi(S,A)\right] - \frac{2\gamma\epsilon}{(1-\gamma)^2}\mathbb{E}_{S\sim d_\pi}\left[D_{\mathsf{TV}}\left(\pi'(\cdot|S), \pi(\cdot|S)\right)\right]$$

- For every π', we have $J(\pi) \geq \mathcal{L}(\pi)$
- If $\pi = \pi'$ then $\mathcal{L}(\pi) = J(\pi)$
- If policy π' is close to π, $\mathcal{L}$ is a good surrogate for J
- **Monotonic Performance Improvement** (Pirotta et al., 2013)

$$\text{If} \qquad \pi^+ \in \arg\max_{\pi'} \mathcal{L}(\pi') \qquad \text{then} \qquad J(\pi^+) \geq J(\pi)$$

- Several approaches proposed based on this surrogate objective (Kakade and Langford, 2002; Wagner, 2011; Pirotta et al., 2013; Schulman et al., 2015; Achiam et al., 2017)

Outline

Towards TRPO

- The direct optimization of $\mathcal{L}(\pi')$ is not trivial
- The **total variation divergence** D_{TV} is hard to optimize (non-differentiable)
  - We bound it with the **KL-divergence** via Pinsker's inequality

$$D_{\text{TV}}\left(\pi'(\cdot|s), \pi(\cdot|s)\right) \leq \sqrt{\frac{1}{2} D_{\text{KL}}\left(\pi'(\cdot|s), \pi(\cdot|s)\right)}$$

- The optimization of the penalized objective is too **conservative**
  - We replace the penalization with a **constraint**

$$\max_{\pi'} \mathcal{L}^{\text{TRPO}}(\pi') := \mathbb{E}_{\substack{S\sim d_\pi \\ A\sim\pi(\cdot|S)}} \left[\frac{\pi'(A|S)}{\pi(A|S)} \mathbb{A}^{\pi}(S, A)\right]$$
$$\text{s.t. } \mathbb{E}_{S\sim d_\pi}\left[D_{\text{KL}}\left(\pi'(\cdot|S), \pi(\cdot|S)\right)\right] \leq \delta$$

where δ is an additional hyperparameter

Trust Region Policy Optimization

- Policies are parametric $\pi_{\boldsymbol{\theta}}(a|s)$ (Schulman et al., 2015)
- The advantage function $\mathbb{A}^{\pi}(s, a)$ is estimated using a **critic** $v_{\mathbf{w}}(s)$
  - Usually with **generalized advantage estimation** (Schulman et al., 2016)
- The **constrained** optimization problem is solved approximately using (conjugate) natural gradient

TRPO Algorithm

Initialize the actor parameters $\boldsymbol{\theta}$ and the critic parameters $\mathbf{w}$

loop for every iteration

Collect a rollout of T total timesteps with $\pi_{\boldsymbol{\theta}}$

Compute value targets y_t and advantage estimates $\widehat{\mathbb{A}}_t$ (e.g. with GAE) based on the current critic $v_{\mathbf{w}}$

Estimate the loss gradient as

$$\mathbf{g} = \sum_{t=0}^{T-1} \nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(A_t|S_t)\widehat{\mathbb{A}}_t$$

Use conjugate gradient to estimate natural gradient $\mathbf{x} = \mathbf{F}(\boldsymbol{\theta})^{-1}\mathbf{g}$

Update the actor parameters $\boldsymbol{\theta}$ via line search

$$\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\sqrt{\frac{2\delta}{\mathbf{x}^{\mathrm{T}}\mathbf{F}(\boldsymbol{\theta})\mathbf{x}}}\mathbf{x}$$

finding the largest α that improves the sample loss and satisfies the (sample) KL-divergence constraint

Define the critic loss

$$\mathcal{L}_{\text{critic}}(\mathbf{w}) = \frac{1}{2}\sum_{t=0}^{T-1}\left(v_{\mathbf{w}}(S_t) - y_t\right)^2$$

Minimize $\mathcal{L}_{\text{critic}}(\mathbf{w})$ w.r.t. $\mathbf{w}$ via gradient descent

end loop

Multi-Epoch TRPO

- TRPO (and the other trust-region methods we will see) are only on-policy if the dataset collected with policy $\pi_{\boldsymbol{\theta}}$ is only used for a single policy update (epoch)
- Multi-Epoch implementations of these algorithms are common
- Multi-Epoch algorithms are off-policy for all but the first epoch of each iteration

Multi-Epoch TRPO

Initialize the actor parameters $\boldsymbol{\theta}$ and the critic parameters $\mathbf{w}$

loop for every iteration

Collect a rollout of T total timesteps with $\pi_{\boldsymbol{\theta}}$

Compute value targets y_t and advantage estimates $\widehat{\mathbb{A}}_t$ (e.g. with GAE) based on the current critic $v_{\mathbf{w}}$

loop for every epoch

Estimate the loss gradient as

$$\mathbf{g} = \sum_{t=0}^{T-1} \nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(A_t|S_t)\widehat{\mathbb{A}}_t$$

Use conjugate gradient to estimate natural gradient $\mathbf{x} = \mathbf{F}(\boldsymbol{\theta})^{-1}\mathbf{g}$

Update the actor parameters $\boldsymbol{\theta}$ via line search

$$\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\sqrt{\frac{2\delta}{\mathbf{x}^{\mathrm{T}}\mathbf{F}(\boldsymbol{\theta})\mathbf{x}}}\mathbf{x}$$

finding the largest α that improves the sample loss and satisfies the (sample) KL-divergence constraint

Define the critic loss

$$\mathcal{L}_{\text{critic}}(\mathbf{w}) = \frac{1}{2}\sum_{t=0}^{T-1}\left(v_{\mathbf{w}}(S_t) - y_t\right)^2$$

Minimize $\mathcal{L}_{\text{critic}}(\mathbf{w})$ w.r.t. $\mathbf{w}$ via gradient descent

end loop

end loop

Outline

From TRPO to PPO

- More **practical** version of TRPO by Schulman et al. (2017)
- Avoid computing the **natural gradient**
- Approximation of the **KL-divergence constraint**
- **First Possibility**: replace the constraint with an **adaptive penalization**

$$\boldsymbol{\theta}_{k+1} \in \arg\max_{\boldsymbol{\theta}\in\mathbb{R}^d} \mathbb{E}_{\substack{S\sim d_{\pi_{\boldsymbol{\theta}}} \\ A\sim\pi_{\boldsymbol{\theta}}(\cdot|S)}} \left[\frac{\pi_{\boldsymbol{\theta}'}(A|S)}{\pi_{\boldsymbol{\theta}}(A|S)}\mathbb{A}^{\pi}(S,A)\right] - \lambda_k \mathbb{E}_{S\sim d_{\pi_{\boldsymbol{\theta}}}}\left[D_{\mathsf{KL}}\left(\pi_{\boldsymbol{\theta}'}(\cdot|S), \pi_{\boldsymbol{\theta}}(\cdot|S)\right)\right]$$

  - Heuristic adaptive λ_k

From TRPO to PPO

- **Second Possibility**: **clipped** objective function
  - Original TRPO **surrogate loss** function

$$\mathcal{L}^{\text{TRPO}}(\boldsymbol{\theta}') = \mathbb{E}_{\substack{S\sim d_{\pi_{\boldsymbol{\theta}}} \\ A\sim\pi_{\boldsymbol{\theta}}(\cdot|S)}}\left[\frac{\pi_{\boldsymbol{\theta}'}(A|S)}{\pi_{\boldsymbol{\theta}}(A|S)}\mathbb{A}^{\pi}(S,A)\right] = \mathbb{E}_{\substack{S\sim d_{\pi_{\boldsymbol{\theta}}} \\ A\sim\pi_{\boldsymbol{\theta}}(\cdot|S)}}\left[\omega_{\boldsymbol{\theta}'/\boldsymbol{\theta}}(S,A)\mathbb{A}^{\pi}(S,A)\right]$$

  - Construct a lower bound by clipping

$$\mathcal{L}^{\text{PPO}}(\boldsymbol{\theta}') \coloneqq \mathbb{E}_{\substack{S\sim d_{\pi_{\boldsymbol{\theta}}} \\ A\sim\pi_{\boldsymbol{\theta}}(\cdot|S)}}\left[\min\left\{\omega_{\boldsymbol{\theta}'/\boldsymbol{\theta}}(S,A)\mathbb{A}^{\pi}(S,A), \text{clip}(\omega_{\boldsymbol{\theta}'/\boldsymbol{\theta}}(S,A), 1-\epsilon, 1+\epsilon)\mathbb{A}^{\pi}(S,A)\right\}\right]$$

- Clipping empirically prevents policy from moving too much away from $\boldsymbol{\theta}$, mitigating both bias (due to $d^{\pi_{\boldsymbol{\theta}}} \neq d^{\pi_{\boldsymbol{\theta}'}}$) and variance (due to policy importance weights)

PPO Algorithm with Clipping

Initialize the actor parameters $\boldsymbol{\theta}$ and the critic parameters $\mathbf{w}$
loop for every iteration
Collect a rollout of T total timesteps with $\pi_{\boldsymbol{\theta}}$
Compute value targets $y_t$and advantage estimates $\widehat{\mathbb{A}}_t$ (e.g. with GAE) based on the current critic $v_{\mathbf{w}}$
Define the

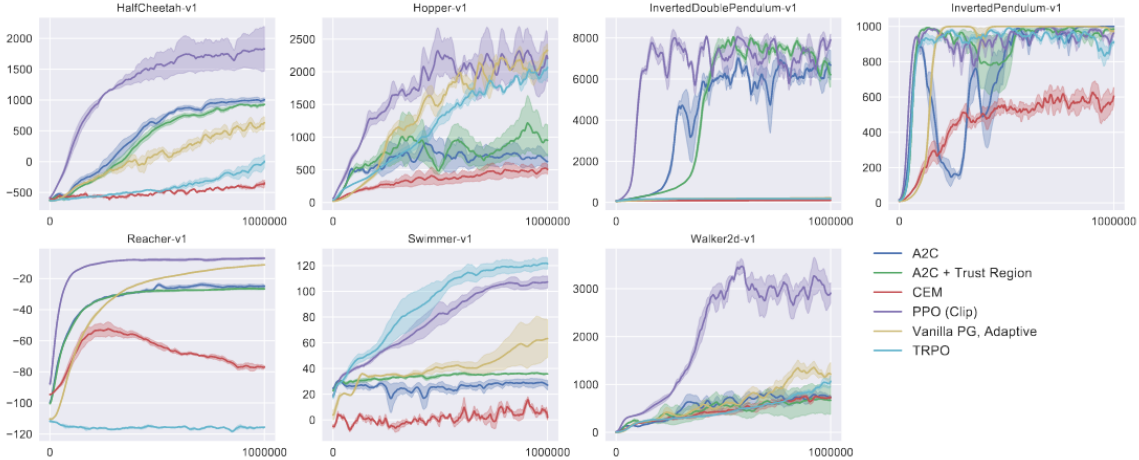
$$\mathcal{L}_{\text{actor}}(\boldsymbol{\theta}) = -\sum_{t=0}^{T-1} \min\left\{\omega_{\boldsymbol{\theta}'/\boldsymbol{\theta}}(S_t^i, A_t^i)\widehat{\mathbb{A}}_t, \text{clip}(\omega_{\boldsymbol{\theta}'/\boldsymbol{\theta}}(S_t^i, A_t^i), 1-\epsilon, 1+\epsilon)\widehat{\mathbb{A}}_t\right\}$$

Define the critic loss

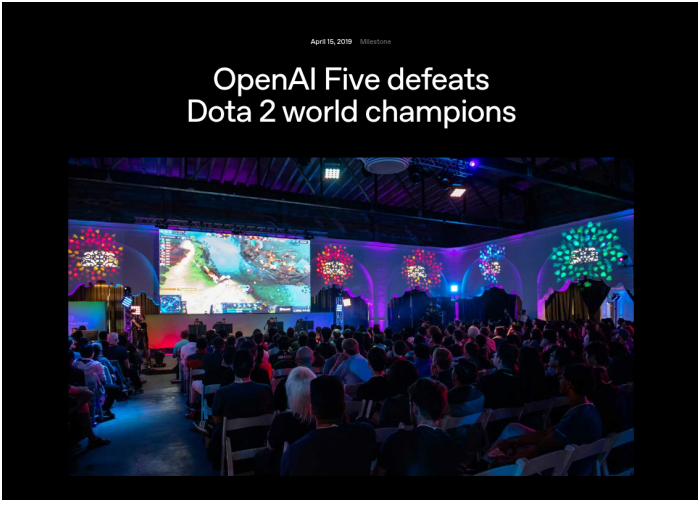
$$\mathcal{L}_{\text{critic}}(\mathbf{w}) = \frac{1}{2}\sum_{t=0}^{T-1}\left(v_{\mathbf{w}}(S_t) - y_t\right)^2$$

Minimize $\mathcal{L}_{\text{actor}}(\boldsymbol{\theta}) + \mathcal{L}_{\text{critic}}(\mathbf{w})$ w.r.t. $\boldsymbol{\theta}$ and $\mathbf{w}$ via gradient descent (one or more epochs)
end loop

PPO - Experimental Results

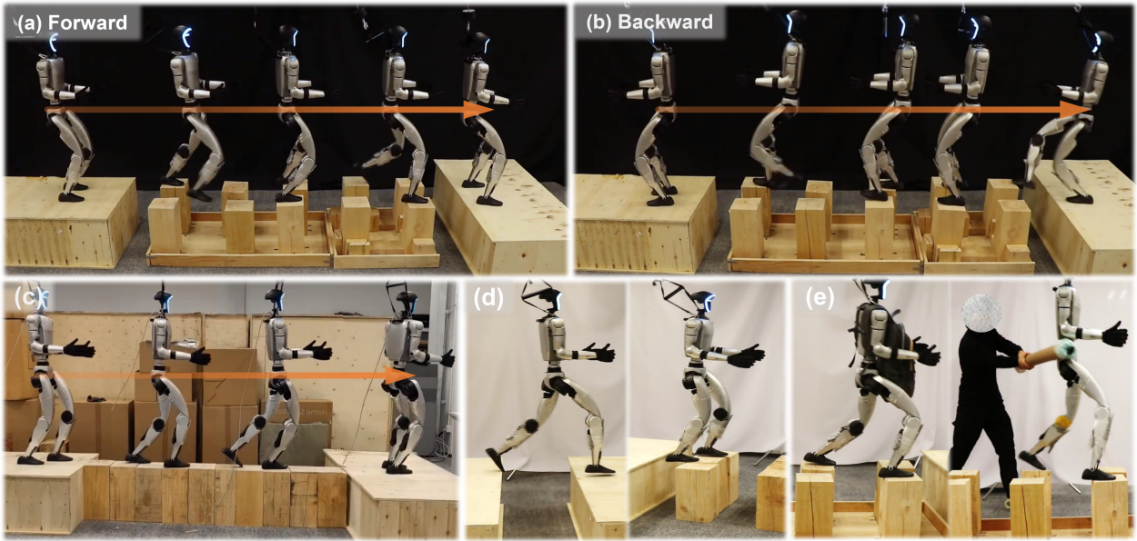


Pictures from (Schulman et al., 2017)



(Berner et al., 2019)

(Wurman et al., 2022)



(Wang et al., 2025)

RLHF
(Reinforcement Learning from Human Feedback)

(Ouyang et al., 2022)

`https://openai.com/index/chatgpt/`

Group Relative Policy Optimization (GRPO)

- **GRPO** is a trust-region method introduced by Shao et al. (2024) for fine-tuning LLMs to solve math problems
- In question answering rewards are sparse (correct/incorrect) and concentrated at the end of the episode (answer) $\implies$ Monte Carlo policy evaluation
- It keeps the clipping of PPO but drops the critic entirely: it's an **actor-only** policy gradient
- A "group" is a batch of trajectories (question-answer pairs)
- **Group-Relative Advantage Estimation** (mean and std over the batch for a fixed timestep t)

$$\overline{R}_t = \frac{R_t - \text{mean}(R_t^i)}{\text{std}(R_t^i)}$$

$$\widehat{\mathbb{A}}_t = \sum_{k \geq t} \overline{R}_{k+1}$$

- Very similar to modern implementations of REINFORCE (Ahmadian et al., 2024)

Algorithm	Class	On-policy	Continuous states	Finite actions	Continuous actions	Deterministic Policy	Reference
DQN	value-based	✗	✓	✓	✗	✓	(Mnih et al., 2015)
REINFORCE	actor-only	✓	✓	✓	✓	✗	(Williams, 1992)
A2C	actor-critic	✓	✓	✓	✓	✗	(Mnih et al., 2016)
DDPG	actor-critic	✗	✓	✗	✓	✓	(Lillicrap et al., 2015)
TD3	actor-critic	✗	✓	✗	✓	✓	(Fujimoto et al., 2018)
SAC	actor-critic	✗	✓	✓†	✓	✗	(Haarnoja et al., 2018)
TRPO	trust-region actor-critic	✓*	✓	✓	✓	✗	(Schulman et al., 2015)
PPO	trust-region actor-critic	✓*	✓	✓	✓	✗	(Schulman et al., 2017)
GRPO	trust-region actor-only	✓*	✓	✓	✓	✗	(Shao et al., 2024)

† See (Christodoulou, 2019)
* Single-epoch version only

Master theses on Reinforcement Learning

- Exploration in policy gradient algorithms
- Multi-agent RL
- Theory of RL with function approximation
- Representation learning and self-supervised RL
- Symbolic RL (reward specification, reasoning...)
- ...

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